NUMERICAL OPTIMISATION CENTRE

The dual reciprocity boundary element method

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Abstract

The Boundary Element Method (BEM) has become an important tool for solving problems in applied science and engineering. Problems may be posed on complex domain. The boundary of the domain often contains corners and points with discontinuous boundary conditions which lead to certain difficulties. The difficulty for problems whose domains contain corners is caused by the ambiguity in the normal derivative at the corners. The difficulty for problems with discontinuous boundary conditions is caused by the fact that the shape functions maintain the continuity of the boundary approximation. A multiple node method is introduced to resolve such problems. A standard BEM program using linear continuous elements has been developed to use the multiple node method and tested on a number of Laplace problems. Computational results are compared with other programs and exact solutions. Errors caused by such problems are substantially reduced.

The boundary element method requires discretisation of the boundary only, thus reducing the quantities of data necessary to run a program. However, there are some difficulties in extending the technique to the non-homogeneous and non-linear case. Such problems require the evaluation of domain integral in the conventional boundary element method. The dual reciprocity method is presented to remove the need of such integrals and two frequently-used radial basis function interpolations are investigated. An existing BEM program is modified to use the dual reciprocity method to solve Poisson problems in which the non-homogeneous term is a function position. This is done by transforming the Poisson problem into a Laplace's problem.

For a problem in which the non-homogeneous term contains the problem variable the transformation into the Laplace problem is not available. A new program based on the dual reciprocity boundary element method is implemented to overcome such problem. The treatment for corners is still needed in the program. Unfortunately, the multiple node method is not suitable to apply with such programs because of the singularity of the approximation matrix. A gradient approach is introduced to use in the program to resolve the corner problems. Non-linear problems require iterative methods and the new program incorporates the Newton technique which has very good convergence properties when a good initial guess is available. Encouraging computer results have been obtained.
Preface

In this report we present the development of a variety of codes to implement the different BEM procedures. The development is indicated in schematic form in Figure 1.

Figure 1  Development of BEM codes

Note: The program in the bracket [...] is implemented to solve all problems on the upper levels
In this work we test the programs on a variety of test problems. The list of these problems relative to the programs is shown in Table 1.

Table 1 The list of test problems

<table>
<thead>
<tr>
<th>Program</th>
<th>LINBEM</th>
<th>MULBEM</th>
<th>GRABEM</th>
<th>MULDRM</th>
<th>DRBEM1</th>
<th>DRBEM2</th>
<th>GRADBEM</th>
<th>GRADBEM (NT)</th>
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<td>Ex.2.3.1</td>
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<td>Ex.3.8.1</td>
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</table>

- Ex.1.4.1 stands for Example 1.4.1 in section 1.4 and so on.
- Ex. stands for Example in section 1.4 and so on.
- ▼ means that the program has been tested on that problem.
- ▲ means that the program is available to solve that problem.

Examples in a same row are identical but tested on different programs. Their numbering are changed to the number of sections which they appeared. The blank cell
means the program in that column is not available to solve the problem in the example in that row. The final program in the last column can solve all problems in the first column.

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