THE LINEAR CITY MODEL

The Example of Choosing only Location without Price Competition

Let \( a \) be the location of firm 1 and \( b \) is the location of firm 2. Assume the linear transporation cost equal to \( td \), where \( t > 0 \). If the good is free, the only cost consumers have to pay is the transportation cost. We have already seen in the midterm that the Nash equilibrium of this model is that both firms are located in the middle of the line and get 50-50 market share.

It is easily seen that the total transportation cost of all people residing in this linear city is \( t/4 \).

Is this a Pareto efficient solution? Of course not. To find the Pareto Optimal solution, the social planner must minimize the transporation cost.

Consider the figure on page 2.

If \( a \) and \( b \) are to be an arbitrary point on the unitary line, we know that \( 0 < a < b < 1 \). Thus,

consumers to the left of \( a \) have the transporation costs equal to \( \frac{1}{2} \times a \times ta = \frac{1}{2}ta^2 \),

consumers between \( a \) and \( \frac{a+b}{2} \) have the transporation costs equal to \( \frac{1}{2} \times \left( \frac{a+b}{2} - a \right) \times t \left( \frac{a+b}{2} - a \right) = \frac{1}{2}t \left( \frac{b-a}{2} \right)^2 \),

consumers between \( \frac{a+b}{2} \) and \( b \) have the transportation costs equal to \( \frac{1}{2}t \left( \frac{a-b}{2} \right)^2 \), and finally,

consumers between \( b \) and 1 have the transporation costs equal to \( \frac{1}{2}t \left( 1 - b \right)^2 \).

Social planner minimizes with respect to \( a \) and \( b \) of the term
This results in two equations and two unknowns so it should be your task to work on to find out the value of $a$ and $b$. The transportation costs of the whole city decrease from $t/4$ to $t/8$. (You should also verify this.)

**The Price Competition Model of Linear City**

There are two firms or stores, which sell the same identical good. For simplicity, suppose that these two stores are located at the extremes of the city; store 1 is at $x = 0$ and store 2 is at $x = 1$. Assume there is no marginal cost with each firm. Consumers incur a transportation cost $t$ per unit of length. Thus, a consumer living at $x$ incurs a cost of $tx$ to go to store 1 and a cost of $t(1 - x)$ to go to store 2. Also assume that each consumer has a unitary demand.

A consumer indifferent between the two firms is located at $x$, where $x$ is demand given by this consumer in equilibrium,

$$p_1 + tx = p_2 + t(1 - x)$$
\[ p_1 + tx = p_2 + t - tx \]

\[ 2tx = p_2 - p_1 + t \]

\[ \therefore x = \frac{p_2 - p_1 + t}{2t}, \quad 1 - x = \frac{p_1 - p_2 + t}{2t}. \]

Suppose we want to find the profit for firm 1. Since by assumption that each consumer has a unitary demand, firm 1 choose price instead of quantity. Hence, firm 1

\[
\max_{p_1} p_1x = \max_{p_1} p_1 \frac{p_2 - p_1 + t}{2t} = \max_{p_1} \frac{p_1 p_2 - p_1^2}{2t} + \frac{p_1}{2}
\]

F.O.C. : \[ \frac{p_2 - 2p_1}{2t} + \frac{1}{2} = 0 \implies p_2 - 2p_1 + t = 0 \]

\[ p_2 = 2p_1 - t. \]

Firm 2 then \( \max_{p_2} p_2 \frac{p_1 - p_2 + t}{2t} \)

F.O.C. : \[ \frac{p_1 - p_2 + t}{2t} = \frac{p_2}{2t} \implies p_1 - p_2 + t = p_2 \implies p_1 + t = 2p_2 \]

\[ p_1 + t = 4p_1 - 2t \implies 3p_1 = 3t \implies \therefore p_1 = p_2 = t. \]

\[ x = \frac{p_2 - p_1 + t}{2t} = \frac{1}{2}, \quad \text{Profit} = \frac{t}{2}. \]

Consider a more generalized version where firm 1 is located at point \( a > 0 \) and firm 2 is located at point \( 1 > b > a \). A rational consumer should be between \( a \) and \( b \), but we are not sure where exactly he is. To determine his location and the firm’s price and profit, we note that in equilibrium, a consumer must be indifferent between the cost obtaining a good so that
\[ p_1 + t (x - a) = p_2 + t (b - x) \]
\[ p_1 + tx - ta = p_2 + tb - tx \]
\[ 2tx = p_2 - p_1 + t (a + b) \]
\[ x = \frac{p_2 - p_1}{2t} + \frac{(a + b)}{2}. \]

Thus, demand for firm 2 is 
\[ 1 - x = 1 - \left( \frac{p_2 - p_1}{2t} + \frac{(a + b)}{2} \right) \]
\[ 1 - x = 1 + \frac{p_1 - p_2}{2t} - \frac{(a + b)}{2} \]

Firm 1 max
\[ \frac{p_1 p_2 - p_1^2}{2t} + \frac{(a + b) p_1}{2} \]
F.O.C. : \[ \frac{p_2 - 2p_1}{2t} + \frac{a + b}{2} = 0 \]
\[ 2p_1 - p_2 = t (a + b) \]
\[ p_2 = 2p_1 - t (a + b) \]

Firm 2 max
\[ \frac{p_1 p_2 - p_2^2}{2t} - \frac{(a + b) p_2}{2} \]
F.O.C. : \[ 1 + \frac{p_1 - 2p_2}{2t} = \frac{a + b}{2} \]
\[ 2t + p_1 - 4p_1 + 2t (a + b) = t (a + b) \]
\[ 2t + t (a + b) = 3p_1 \]
\[ \therefore p_1 = \frac{t (2 + a + b)}{3}, p_2 = \frac{2t (2 + a + b)}{3} - t (a + b) = \frac{4}{3} t - \frac{1}{3} t (a + b) \]

Since both firms have zero marginal cost, it must be that firm 1 is getting demand equal to that of firm 2. This means \( a = 1 - b \).

Hence, \( p_1 = p_2 = t, x = \frac{1}{2}, \pi = \frac{t}{2} \)
THE CIRCULAR CITY MODEL

Consider the circular city where consumers are located uniformly on a circle with a perimeter equal to 1. Density is unitary around the city, and all consumers travel along the perimeter of the circle (not across the circle). Think of the city around the circular lake, for example.

We still retain the assumption that each consumer buys one unit of the good, has a unit transport cost $t$, and is willing to buy at the smallest transportation cost. The firms have zero marginal cost.

Let $n$ denote the number of entering firms. Those firms do not choose their locations, but rather are automatically located equidistant from one another on the circle. Then firms compete in prices given these locations. Also assume free entry.

Since $n$ firms entered the market and were located symmetrically around the circle, it is reasonable to consider an equilibrium in which they all charge the same price $p$ since the good is identical. In practice, we can consider only firm $i$ and it has only two real rivals; namely those around it. Consider the next figure.

In equilibrium, a consumer located at the distance $x \in (0, \frac{1}{n})$ from firm $i$ must be indifferent between purchasing from firm $i$ and purchasing from another firm besides $i$. Hence,

$$p_i + tx = p_j + t \left( \frac{1}{n} - x \right) \rightarrow p_i + tx = p_j + \frac{t}{n} - tx \rightarrow 2x = \frac{p_j - p_i}{t} + \frac{1}{n},$$
Note that $2x$ is the demand for each firm since one $x$ accounts for the left hand side and another $x$ accounts for the right hand side.

Therefore, firm $i$ max $p_i \left[ \frac{p_j - p_i}{t} + \frac{1}{n} \right] = \max p_i \frac{p_ip_j - p_i^2}{t} + \frac{p_i}{n}$.

F.O.C. : $\frac{p_j - 2p_i}{t} + \frac{1}{n} = 0 \Rightarrow 2p_i - p_j = \frac{1}{n} \rightarrow 2p_i - p_j = \frac{t}{n}$.

Since the model is symmetric, you can show that we will derive another equation, namely $2p_j - p_i = \frac{t}{n}$. Therefore, in equilibrium, $p_i = p_j = \frac{t}{n}$.

Each consumer should be located at the distance $x^* = \frac{1}{2n}$. We can see that each firm’s profit decreases with the number of firms.

If we are to impose the fixed cost in entering the market for each firm, say $f$. With free entry and exit, we can determined the equilibrium number of firms as total zero profit condition.

$$\frac{t}{n^2} - f = 0 \Rightarrow n^2 = \frac{t}{f} \Rightarrow n_{eq} = \sqrt{\frac{t}{f}}.$$  

With higher fixed cost, the total number of firms will decrease.

Another point to address is the socially optimal number of firms. Is $n_{eq}$ a socially optimum? Is $n_{eq}$ too much or too little a number of firms? We need to consider the social welfare of the city.

We see that the total cost of traveling is the area of the triangle as in the first example. Each triangle has the area of $\frac{1}{2} \times x^* \times \frac{t}{2n} = \frac{1}{2} \times \frac{1}{2n} \times \frac{t}{2n}$.  

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But we have totally \( 2n \) triangles (why?) Thus, the total cost of traveling is \( \frac{t}{4n} \). The social planner minimizes the total transportation costs plus total fixed costs.

\[
\min_n \frac{t}{4n} + nf 
\]

\text{F.O.C.: } f = \frac{t}{4n^2} \rightarrow n^2 = \frac{t}{4f} \rightarrow n_{\text{optimal}} = \frac{1}{2} \sqrt{\frac{T}{f}} < n_{eq}.

Note that counting the number of triangle like the first example can be done with only linear transportation cost. If the cost is not linear; i.e., the general function of \( x \), namely, \( f(x) \). The transportation cost must be the

\[2nt \int_0^{\frac{\pi}{2}} f(x) \, dx,\]

so the social planner should minimize \( nf + 2nt \int_0^{\frac{\pi}{2}} f(x) \, dx\).

A trivial point but of some importance of this circular model is that firms price above marginal cost and yet do not make profits. Thus an empirical finding that firms do not make more than normal profits in an industry should not lead one to conclude that firms do not have market power, where market power is defined as pricing above marginal cost.